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Graph manifolds with boundary are virtually special

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Abstract

Let M be a graph manifold. We prove that fundamental groups of embedded incompressible surfaces in M are separable in $\pi_1(M)$, and that the double cosets of crossing surfaces are also separable. We deduce that if there is a "sufficient" collection of surfaces in M, then $\pi_1(M)$ is virtually the fundamental group of a "special" nonpositively curved cube complex. That is a complex that admits a local isometry into the Salvetti complex of a right-angled Artin group. We provide a sufficient collection for graph manifolds with boundary thus proving that their fundamental groups are virtually special, in particular linear.

1 Introduction

A *graph manifold* is an oriented compact connected 3–manifold whose prime summands have only Seifert pieces in their JSJ decompositions. Hempel has proved [Hem87, Theorem 1.1] that the fundamental groups of all Haken 3–manifolds, in particular all graph manifolds, are residually finite. Throughout the article we assume that a graph manifold does not admit a prime summand which is a single Seifert fibered space. For background on graph manifolds we refer the reader to the survey article by Buyalo and Svetlov [BS04].

We are interested in separability properties of surfaces embedded in graph manifolds. We say that a subgroup F of a group G is *separable* if for each $g \in G \setminus F$, there is a finite index subgroup H of G with $g \notin HF$.

Let S be an oriented incompressible surface embedded in a graph manifold M. Then, by the loop theorem, $\pi_1(S)$ embeds in $\pi_1(M)$. Let us elaborate on

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the viewpoint on the fundamental groups we take in this article. By $\pi_1(M)$ we denote the group of covering transformations of the universal cover $\widetilde{M}$ of M. Once we choose a basepoint $\tilde{m} \in \widetilde{M}$ we can assign to a covering transformation g a path $\widetilde{\gamma}$ joining $\tilde{m}$ to $g\tilde{m}$. This path projects to a loop in M based at the projection m of $\tilde{m}$. This allows us to identify the group of covering transformations $\pi_1(M)$ with the fundamental group of (M, m) and justifies the notation $\pi_1(M)$.

An *elevation* $S' \to M'$ of an embedding $S \to M$ is an embedding of a cover S' of S into a cover M' of M which covers $S \to M$. (A *lift* is an elevation with $S' = S$.) Each elevation $\widetilde{S} \subset \widetilde{M}$ of $S \subset M$ determines an embedding of $\pi_1(S)$ into $\pi_1(M)$ ($\pi_1(S)$ stabilises $\widetilde{S} \subset \widetilde{M}$). Embeddings of $\pi_1(S)$ coming from different elevations differ by conjugation in $\pi_1(M)$. We fix one such embedding $\pi_1(S) \subset \pi_1(M)$. We prove:

Theorem 1.1. *Let M be a graph manifold (possibly without boundary). Let S be an oriented incompressible surface embedded in M. Then $\pi_1(S)$ is separable in $\pi_1(M)$.*

Note that a subgroup is separable if any of its conjugates is separable, so the above statement does not depend on the choice of the embedding $\pi_1(S) \subset \pi_1(M)$.

More generally, let $F_1, F_2 \subset G$ be two subgroups of a group G. We say that the *double coset* F_1F_2 is *separable*, if for each $g \in G \setminus F_1F_2$ there is a finite index subgroup H of G with $g \notin HF_1F_2$.

Now let S and P be oriented incompressible surfaces embedded in a graph manifold M. Assume that the intersections of S and P with each block of M are horizontal or vertical (for definitions see Section 2, this can be achieved via a homotopy). Let $\pi_1(S), \pi_1(P) \subset \pi_1(M)$ be embeddings coming from specified elevations $\widetilde{S}, \widetilde{P} \subset \widetilde{M}$. We say that $\pi_1(S), \pi_1(P)$ are *crossing* if $\widetilde{S}$ and $\widetilde{P}$ intersect. Note that this property is not invariant under homotopy. Our main theorem is the following:

Theorem 1.2. *Let M be a graph manifold. Let S, P be oriented incompressible surfaces embedded in M with crossing $\pi_1(S), \pi_1(P) \subset \pi_1(M)$. Then $\pi_1(S)\pi_1(P)$ is separable in $\pi_1(M)$.*

We apply Theorems 1.1 and 1.2 to obtain:

Theorem 1.3. *Let M be a graph manifold with boundary. Then $\pi_1(M)$ is virtually a fundamental group of a special cube complex.*

A *special* cube complex is a nonpositively curved cube complex which admits a local isometry into the Salvetti complex of a right-angled Artin group (for more information, see [HW08] and [HW10]). As a consequence, the fundamental groups of special cube complexes (which will be called *special* also) are subgroups of right-angled Artin groups. The latter have various outstanding properties. To mention just a few, they are linear over $\mathbf{Z}$ [Hum94] and residually torsion-free nilpotent [DK92]. Moreover, they virtually satisfy Agol's RFRS condition [Ago08].

Wise has proved that fundamental groups of closed hyperbolic 3–manifolds with an embedded geometrically finite surface are virtually special [Wis11]. A relative version of this theorem says that the fundamental groups of all finite volume hyperbolic manifolds with boundary are virtually special as well. Our theorem treats (partially) the complementary case, with an eye on analysing in future the case of manifolds with both hyperbolic and Seifert fibered pieces.

The class of graph manifolds with boundary has been studied by Wang and Yu who prove [WY97, Theorem 0.1] that they all virtually fiber over the circle. (Note that we do not exploit that result in our article.) A closed graph manifold might not virtually fiber [LW97]. Hence, by Agol's virtual fibering criterion [Ago08] such a manifold cannot have a virtually special fundamental group. That explains that some hypothesis is needed in Theorem 1.3.

In fact, we have recently learned that independently Yi Liu has proved [Liu11, Theorem 1.1] that the graph manifolds with virtually special fundamental groups are exactly the ones that admit a nonpositively curved Riemannian metric. It was proved by Leeb [Lee95, Theorem 3.2] that graph manifolds with boundary admit a nonpositively curved Riemannian metric (with geodesic boundary). Hence our Theorem 1.3 is a special case of the theorem of Liu.

In order to obtain Theorem 1.3 we prove, using Theorems 1.1 and 1.2, the following criterion involving "sufficient" collections of surfaces. (For definitions see Section 2.)

Definition 1.4. Let $\mathcal{S}$ be a collection of incompressible oriented surfaces embedded in a graph manifold M satisfying the property that the intersection of each surface from $\mathcal{S}$ with each block of M is vertical or horizontal. We say that $\mathcal{S}$ is *sufficient* if

(1) for each block $B \subset M$ and each torus $T \subset B$ in its boundary, there is a surface $S \in \mathcal{S}$ such that $S \cap T$ is non-empty and vertical w.r.t. B,

(2) for each block $B \subset M$ there is a surface $S \in \mathcal{S}$ such that $S \cap B$ is horizontal.

Note that property (1) automatically implies property (2). Indeed, let B_0 be a block and let B_1 be any adjacent block. Put $T = B_0 \cap B_1$. By (1) there is a surface S such that $S \cap T$ is vertical in B_1. Then $S \cap B_0$ is horizontal.

Our criterion is the following:

Theorem 1.5. *Assume that a graph manifold M admits a sufficient collection $\mathcal{S}$. Then $\pi_1(M)$ is virtually special.*

Once we prove Theorem 1.5, in order to derive Theorem 1.3 it remains to construct a sufficient collection for graph manifolds with boundary.

Note that in course of his proof [Liu11, Lemma 4.7] Liu constructs a set of cohomology classes giving rise to a sufficient collection. Hence combining this with Theorem 1.5 one can get an alternate (but more complicated) argument for Liu's theorem that all graph manifolds admitting a nonpositively curved Riemannian metric have virtually special fundamental groups. Liu also suggested to us that cut and paste operations on the surfaces obtained in [BS04, Section 5.5.3] yield a sufficient collection as well.

The article is organised as follows. In Section 2 we discuss notation. In Section 3 we derive Theorem 1.3 from Theorems 1.1 and 1.2. More precisely, we first prove Theorem 1.5 and then prove that graph manifolds with boundary have virtually a sufficient collection (Proposition 3.1). In Section 4 we prepare the background for the proofs of Theorem 1.1 in Section 5 and Theorem 1.2 in Section 6.

Acknowledgements

We thank Yi Liu for sharing with us his article [Liu11], and for providing feedback on our preprint leading to many improvements.

2 Notation

A graph manifold will be denoted by M. We can and will assume that M is irreducible. The JSJ tori decompose M into pieces which will be called *blocks* (denoted usually by M_v or B). By passing to a finite index cover [RW98, Lemma 1.10] we can assume that all the blocks are products $M_v = S^1 \times F_v$, where F_v is an oriented surface (with boundary) of nonzero genus. Then M is called *simple*. The induced quotient map $\pi_1(M_v) \to \pi_1(F_v)$ does not depend on the choice of the product structure.

Let S be an oriented surface embedded in M. Then $\pi_1(S)$ embeds in $\pi_1(M)$ by the loop theorem. Throughout the article, the surface S (or P) will always be assumed to be oriented. The surface S can be homotoped so

that each component of $S \cap M_v$ (called a *piece*) is either *vertical* (fibered by S^1 and essential) or *horizontal* (transverse to the fibers thus covering F_v). Since S is embedded, for each block M_v either components of $S \cap M_v$ are all horizontal or all vertical, unless $S \cap M_v$ is empty. In that case we call the block *S–horizontal, S–vertical* or *S–empty*, accordingly. When the surface S in question is understood, we simply call the block *horizontal, vertical* or *empty*.

We will be also considering possibly noncompact covers $M' \to M$ of graph manifolds. The connected components in M' of the preimage of blocks of M will be also called *blocks*. Also, if a specified elevation of S crosses a block $M'_v \to M_v$, then this block will be called *horizontal* or *vertical* if M_v is such. Other blocks of M' will be called *empty*.

3 Cubulation

Proof of Theorem 1.5. Complete the collection $\mathcal{S}$ to $\mathcal{S}'$ by adding all JSJ tori, and vertical tori in each block M_v whose base curves on the surface F_v *fill* F_v. This means that the complementary regions of the union of the geodesic (in some hyperbolic metric on F_v) representatives of the base curves are discs or once-punctured discs. Note that if we add to that family of curves the base arcs of the annuli guaranteed by property (1) of a sufficient collection, all the complementary regions become discs. Call such a family *strongly filling*.

After a homotopy we can assume that all the surfaces in $\mathcal{S}'$ are pairwise transverse. Each elevation of an incompressible surface from $\mathcal{S}'$ to the universal cover of M splits it (up to a set of measure 0) into two components. This gives the universal cover of M the structure of a "space with walls" (see [CN05] or [Nic04]). We can consider the action of $\pi_1(M)$ on the associated Sageev CAT(0) cube complex X (also see [CN05]).

We claim that the group $\pi_1(M)$ acts freely on X. To justify that, pick $g \in \pi_1(M)$. If g does not lie in $\pi_1(B)$ for some block B, then it acts freely on the underlying graph of the graph manifold structure of the universal cover of M. Hence g also acts freely on X, since we have included the JSJ tori in $\mathcal{S}'$. Otherwise, suppose $g \in \pi_1(B)$. If g is not central ("not vertical") in $\pi_1(B)$, then by the strong filling property for the vertical pieces of B, g acts freely on the tree associated with one of the pieces. Consequently, it acts freely on X. Otherwise, g is central in $\pi_1(B)$ and the claim follows from the existence of a horizontal piece in B among the surfaces in $\mathcal{S}'$ (property (2) of a sufficient collection). Note that the action of $\pi_1(M)$ on X might not be and usually is not cocompact.

We now invoke [HW10, Theorem 4.5], which is a criterion for $X/\pi_1(M)$

to be virtually special. In the case of the cube complex X arising from a collection $\mathcal{S}'$ of compact π_1–injective surfaces in a 3–manifold M for this criterion to be satisfied it is enough to have:

(1) $\mathcal{S}'$ is finite,

(2) for each surface $S \in \mathcal{S}'$, in the $\pi_1(S)$ cover $M^S = \widetilde{M}/\pi_1(S)$ of M there are only finitely many elevations of surfaces in $\mathcal{S}'$ disjoint from S, but not separated from S by another elevation of a surface from $\mathcal{S}'$,

(3) for each $S \in \mathcal{S}'$ the subgroup $\pi_1(S)$ is separable in $\pi_1(M)$,

(4) for each pair of intersecting elevations of $S, P \in \mathcal{S}'$, their corresponding double coset $\pi_1(S)\pi_1(P)$ is separable in $\pi_1(M)$.

Condition (1) is immediate, conditions (3) and (4) are supplied by Theorems 1.1 and 1.2. It remains to discuss condition (2):

Fix $S \in \mathcal{S}'$ and let $\widetilde{P}$ be an elevation of a surface in $\mathcal{S}'$ to the $\pi_1(S)$ cover M^S of M. Assume that $\widetilde{P}$ is disjoint from (the lift of) S but not separated from S by another elevation of a surface from $\mathcal{S}'$. Then $\widetilde{P}$ must intersect at least one (of the finitely many) blocks B of M^S intersecting $S \subset M^S$ (otherwise an elevation of a JSJ torus separates S and $\widetilde{P}$). We fix the block B. Assume first that $\widetilde{P} \cap B$ is horizontal and that a component of $\widetilde{P} \cap B$ projects to a specified piece of the finitely many pieces of $\mathcal{S}'$. Then there can be at most 2 such $\widetilde{P}$, since they are all nested.

Now assume that $\widetilde{P} \cap B$ is vertical. Consequently, $S \cap B$ is also vertical. Then the whole configuration can be analysed using the base curves on the base surface F of B. For a strongly filling family of curves, each pair of their elevations to the universal cover of F not separated by a third one has to be at a uniformly bounded distance. Hence also in this case there are only finitely many possible P. This concludes the argument for condition (2).

Hence all the conditions of the Haglund–Wise criterion are satisfied and the cube complex $X/\pi_1(M)$ is virtually special. □

As an application, we will consider graph manifolds with boundary.

Proposition 3.1. *A graph manifold with boundary admits a finite cover with a sufficient collection.*

Presently, observe that Proposition 3.1 and Theorem 1.5 readily yield Theorem 1.3.

In the proof of Proposition 3.1 we will need the following:

Lemma 3.2 (a version of [WY97, Lemma 1.1]). *Let $T_1, \ldots, T_n$ be the boundary components of a block M_v. Assume we are given families of disjoint identically oriented circles $C_1 \subset T_1, \ldots, C_{n-1} \subset T_{n-1}$ such that the oriented intersection number between C_i and the vertical fiber is non-zero and does not depend on $1 \leq i \leq n-1$. Then there is a family of disjoint identically oriented circles $C_n \subset T_n$ such that the union of C_i over all $1 \leq i \leq n$ is the boundary of an oriented horizontal surface embedded in M_v.*

Proof of Proposition 3.1. Let Γ be the underlying graph of the graph manifold M. A vertex w of Γ is called a *boundary* vertex, if its block M_w has a torus boundary component contained in the boundary of M. Note that if M has boundary, then there exists a boundary vertex.

We first pass to a finite cover of M which is simple (see Section 2). Moreover, we want to pass to a finite cover whose underlying graph Γ has the following property.

(antennas) For each pair of adjacent vertices $v_0, v_1 \in \Gamma$ there is an embedded edge-path $(v_0, v_1, v_2, \ldots, v_n)$ such that

(i) the subpath $(v_1, v_2, \ldots, v_n)$ is a full subgraph (i.e. induced subgraph) of Γ, and

(ii) v_n is a boundary vertex.

We will first construct a sufficient collection using (antennas) property and later justify how to pass to a cover satisfying (antennas).

As discussed in the introduction, it is enough to obtain property (1) of a sufficient collection. Let $B = M_{v_0}$ be a block and let T be a torus in its boundary. Let C_0 be the vertical w.r.t. B circle on T. If T is a boundary torus of the whole M, then we put $n = 0$, otherwise we define v_1 so that M_{v_1} is the block distinct from M_{v_0} containing T. Then applying (antennas), we obtain an edge-path satisfying (i) and (ii). We will find a properly embedded surface S_n intersecting T along circles in the direction of C_0.

For $i = 0$ to n we inductively define surfaces $S_0 \subset \ldots \subset S_i \subset \ldots \subset S_n$ embedded in M, but not necessarily properly: S_i might have boundary components in $M_{v_i} \cap M_{v_{i+1}}$.

We define the surface S_0 to be the vertical annulus in $B = M_{v_0}$ joining T to itself, not separating M_{v_0} (uses the fact that M is simple). If $n = 0$, then we are done.

Otherwise, let $i \geq 1$ and assume that S_{i-1} has already been constructed, but is not proper. Let C_{i-1} denote one of the boundary circles of S_{i-1} in $M_{v_{i-1}} \cap M_{v_i}$. If C_{i-1} is vertical in M_{v_i}, that means that we can complete S_{i-1} immediately to S_n by adding several vertical annuli in M_{v_i}.

Otherwise, let $\mathcal{E}$ be the family of all edges adjacent to v_i, distinct from the edges joining it to v_{i-1} and v_{i+1} (if it is defined). Since by (antennas) property the path $(v_j)_{j=1}^n$ is full, all the edges in $\mathcal{E}$ join v_i to a vertex outside the path $(v_j)_{j=1}^n$. Similarly as we have done for the edge (v_1, v_0), for each edge $e = (v_i, w) \in \mathcal{E}$ we take a vertical annulus A_e in M_w joining the boundary torus of M_w corresponding to e to itself. Again we require that A_e does not separate M_w and because of that we can take it disjoint from all the annuli in M_w constructed for smaller values of i, assigned to other boundary components (see Figure 1).

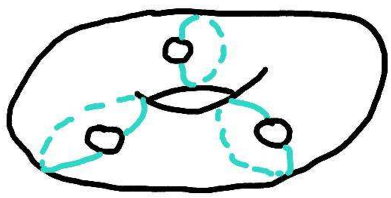

Figure 1: bases of A_e in common M_w from different i

The annuli A_e specify circles C_e on the tori $M_w \cap M_{v_i}$. At this point on all but one (or all if $i = n$) boundary tori of M_{v_i} which are not in the boundary of M we have constructed non-vertical circles C_{i-1} or C_e. For all the boundary tori K of M in M_{v_i}, except for one (call it Q) in the case where $i = n$, we pick arbitrary horizontal circles C_K.

By Lemma 3.2, if we take appropriate orientations on the circles C_{i-1}, C_e, C_K and we take appropriately many copies, we can find an oriented circle C_i on the remaining boundary torus of M_{v_i} (connecting to $M_{v_{i+1}}$, or being Q), such that appropriately many copies of C_i together with the other Cs span an embedded horizontal surface H_i.

Taking appropriately many copies of A_e, S_{i-1} and H_i, every other carrying an opposite orientation, we form the surface S_i.

Inductively, we arrive at the required surface S_n needed for property (1) of a sufficient collection. See Figure 2.

It remains to explain how to obtain the property (antennas). The fullness part is automatic if

- Γ has no double edges and edges joining a vertex to itself (which we arrange using residual finiteness of the free group $\pi_1(\Gamma)$) and
- $(v_i)_{i=1}^n$ is always chosen to be geodesic.

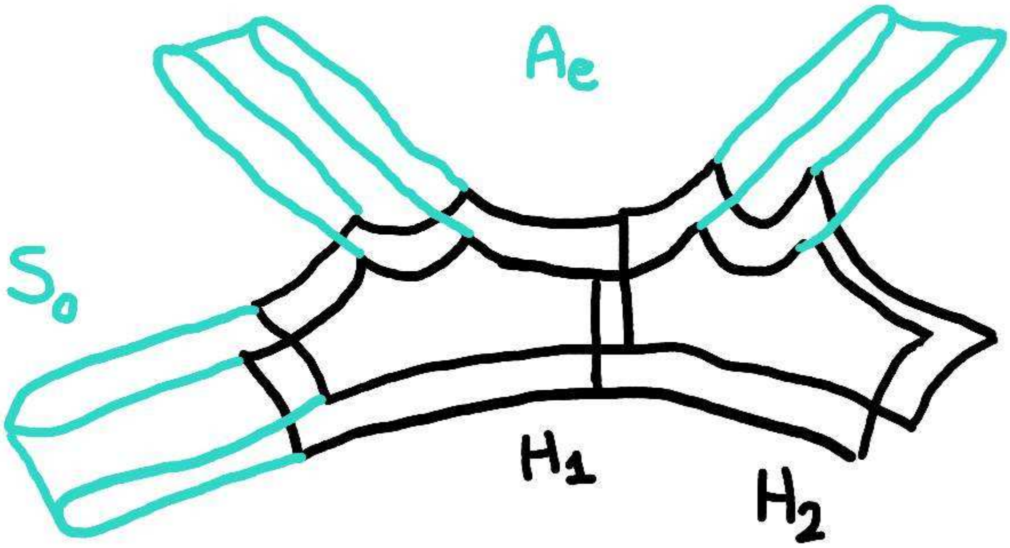

Figure 2: the surface S_2

In view of this, it suffices to pass to Γ such that for each vertex v_1 there is a geodesic terminating at a boundary vertex v_n which does not pass through a prescribed neighbor v_0 of v_1.

To do this, we take the following cover of M of degree 2^k, where k is the number of blocks of M. The cover is defined by the mapping of $H_1(M, \mathbf{Z})$ into $\mathbf{Z}_2^k$ determined by the cohomology classes of closed non-separating vertical tori, one in each of the k blocks. Denote by Γ' the underlying graph of the cover graph manifold. See Figure 3.

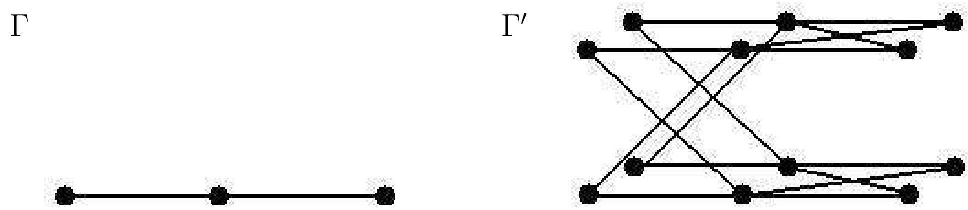

Figure 3: graphs Γ and Γ'

Fix vertices $v_0, v_1 \in \Gamma'$ and let γ be a geodesic path in Γ' from v_1 to a boundary vertex v_n. In the case where γ goes through v_0, we alter it:

Consider the copy of $\mathbf{Z}_2$ in the group of covering transformations (acting also on Γ') corresponding to the block covered by v_1. Then the nontrivial element of that $\mathbf{Z}_2$ maps γ to a geodesic path disjoint from v_0 terminating at a boundary vertex. This shows that our degree 2^k cover satisfies (antennas) and closes the proof of Proposition 3.1. $\square$

4 Separability: preliminaries

In this section we prepare the background for the proofs of Theorems 1.1 and 1.2.

Hempel's theorem

We begin with discussing lemmas related to Hempel's theorem:

Theorem 4.1 (special case of [Hem87, Theorem 1.1])**.** *Fundamental groups of graph manifolds are residually finite.*

In the proof Hempel uses a criterion ([Hem87, Theorem 3.1]) whose verification requires more: separability in $\pi_1(M)$ of the boundary tori subgroups from the block groups:

Lemma 4.2. *Let B be a block of a graph manifold M and T a torus in the boundary of B. Let $g \in \pi_1(B) \subset \pi_1(M)$ be an element outside $\pi_1(T) \subset \pi_1(B)$. Then there is a finite index subgroup H of $\pi_1(M)$ with $g \notin H\pi_1(T)$.*

Geometrically, the lemma says the following. Assume the basepoint of all the spaces is in T. Let M_H be the H cover of M. Let γ be the lift of the path representing $g \in \pi_1(B)$ starting at the basepoint of M_H. Then γ does not end on the elevation of T through the basepoint. The proof follows from [Hem87, Lemma 4.1] in the same way as Theorem 4.1 does (page 388 of [Hem87]) and we omit it. We will use several consequences of Lemma 4.2.

Corollary 4.3. *Up to passing to a finite cover we can always assume that a surface S in a graph manifold M is* straight*, i.e. its vertical annular pieces always join two distinct boundary components of the block.*

The next corollary follows from Lemma 4.2 and the residual finiteness of the free group $\pi_1(\Gamma)$, where Γ is the underlying graph of the graph manifold M.

Corollary 4.4. *Let γ be a path in a graph manifold M such that its lift $\widetilde{\gamma}$ to the universal cover $\widetilde{M}$ goes through as few blocks of $\widetilde{M}$ as possible in its homotopy class (fixing the endpoints). Then there is a finite cover M' of M, where γ' (the quotient of $\widetilde{\gamma}$) does not go twice through the same JSJ torus.*

Combining Corollary 4.4 with Lemma 4.2 we obtain a special case of Theorem 1.1.

Corollary 4.5. *If T is a torus in a boundary of a block of a graph manifold M, then $\pi_1(T)$ is separable in $\pi_1(M)$. More generally, this is true for any vertical torus T in some block of M (since one can add it artificially to the JSJ collection).*

Untwining arcs

The following result will play an important role in the proof of Theorem 1.2.

Proposition 4.6. *Let F be a surface with two distinguished boundary components C_1, C_2 joined by an embedded arc α. Let $\mathcal{A}$ be a finite family of arcs properly embedded in F with endpoints on C_1 and C_2. Then for each n sufficiently large there is a cover $\widetilde{F}$ of F of degree $n!$ on each boundary component and satisfying the following.*

(*) *There is a lift $\tilde{\alpha}$ of α such that, if $\widetilde{C}_i$ are the elevations of C_i through the endpoints of $\tilde{\alpha}$, then all of the lifts of the arcs in $\mathcal{A}$ starting from $\widetilde{C}_i$ do not terminate at $\widetilde{C}_j$, except possibly for arcs homotopic to $\tilde{\alpha}$ relative the boundary circles (if $\mathcal{A}$ contains arcs homotopic to α).*

In the proof, we will need the following "omnipotency" lemma.

Lemma 4.7. *Let F be a surface of non-zero genus. Assign a number $n_i > 0$ to each boundary component C_i of F. Then there is a finite cover $\widetilde{F}$ of F of degree n_i on each component of the preimage of C_i.*

Proof. Since F has non-zero genus, there is a non-separating simple closed curve $\beta \subset F$. Take the double cover F_d determined by the cohomology class $[\beta] \in H^1(F, \mathbf{Z}_2)$. Each boundary component C_i of F lifts now to a pair of boundary components C_i^1, C_i^2 of F_d. Now pick a family of disjoint simple arcs β_i joining C_i^1 to C_i^2. Take the cover $\widetilde{F}$ determined by the mapping of $H_1(F_d, \mathbf{Z})$ to $\prod \mathbf{Z}_{n_i}$ determined by the cohomology classes $[\beta_i] \in H^1(F_d, \mathbf{Z}_{n_i})$. □

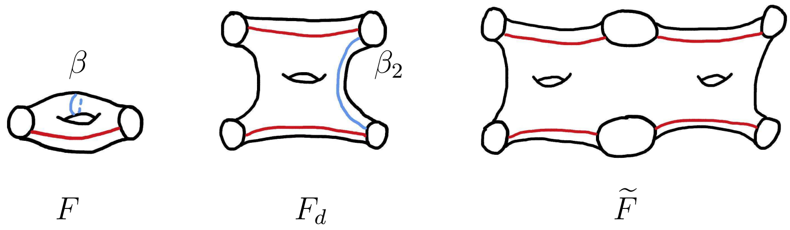

Figure 4: β, β_2 and the lifts of α to F_d and $\widetilde{F}$ with $d_1 = 1, d_2 = 2$

Remark 4.8. There is an extra feature to the construction in the proof of Lemma 4.7. If $\alpha \subset F$ is an arc joining two distinct boundary components of F, then no two lifts of α to $\widetilde{F}$ join the same pair of boundary components. See Figure 4.

Proof of Proposition 4.6. First we find $\widetilde{F}$ satisfying property (*) but having possibly wrong degree on the boundary components. We consider $H = \pi_1(C_1 \cup \alpha \cup C_2) \subset \pi_1(F)$. Since H is a finitely generated subgroup of the free group $\pi_1(F)$, it is separable. We can assume that all the endpoints of the arcs in $\mathcal{A}$ actually coincide with the endpoints of α. Let $A \subset \pi_1(F)$ be the finite set of elements determined by $\alpha \cup \alpha_i^{-1}$, for those arcs $\alpha_i \in \mathcal{A}$ which are not homotopic to α relative to the boundary circles. Note that the set A is disjoint from H. Hence there is a finite index subgroup G of $\pi_1(F)$ containing H but none of the finitely many elements of A. The cover $\widetilde{F}$ corresponding to G satisfies (*).

Let n be so large that $n!$ is divisible by all the degrees on the boundary components of the cover $\widetilde{F} \to F$. We now pass from $\widetilde{F}$ to a further cover having degrees $n!$ on the boundary, using Lemma 4.7. Note that if $\mathcal{A}$ does not contain an arc homotopic to α, then passing to a finite cover we do not loose (*), and we are done. Otherwise, if α lies in $\mathcal{A}$, we have to additionally invoke Remark 4.8. $\square$

Surface-injective covers

In the proof of Theorems 1.1 and 1.2 we will need so called "surface-injective" covers. As a preparation, we discuss the structure of the following infinite cover. As usual, we assume that $S \subset M$ is an incompressible oriented surface embedded in a graph manifold M, and that the intersections of S with all the blocks of M are vertical or horizontal.

Definition 4.9. We denote by M^S the (infinite) cover $\widetilde{M}/\pi_1(S)$ of S corresponding to $\pi_1(S) \subset \pi_1(M)$. Let us describe the topology of non-empty blocks of M^S. For each horizontal component S_0 of $S \cap M_v$, there is in M^S an associated horizontal block $M_v^{S_0} \cong S_0 \times \mathbf{R}$. Similarly, for each vertical component S_0 of $S \cap M_v$, there is in M^S an associated vertical block $M_v^{S_0} \cong S^1 \times \widetilde{F}_v$. The annulus S_0 embeds inside $M_v^{S_0}$ as a product of the factor S^1 and a proper arc on $\widetilde{F}_v$. All boundary components of the vertical and horizontal blocks of M^S are $S^1 \times \mathbf{R}$.

Definition 4.10. Let S be a surface in M. A finite cover M' of M (possibly $M' = M$) is called *S–injective*, if S lifts to M' and for each block B' of M', the surface $S \cap B'$ is connected. Moreover, we require that each horizontal component of $S \cap B'$ maps with degree 1 onto the base surface of B'.

In particular, M' arises from M^S by quotienting each non-empty block separately (though empty blocks are identified). Note that a priori it is not immediate that there exist S–injective covers of M. Moreover, the property of being S–injective is not preserved under passing to a further cover. In a moment we will provide a construction of (high-degree) S–injective covers. Before we give the construction, we need the following terminology and lemma.

Definition 4.11. A *partial* finite cover of a graph manifold M is a graph manifold $\overline{M}$ together with a map $\overline{M} \to M$ which is a cover except possibly on the components of the boundary which are mapped to JSJ tori. These boundary components are called *cut tori.*

Lemma 4.12. *Assume we have a partial finite cover $p\colon \overline{M} \to M$. Assume also that all cut tori of $\overline{M}$ map homeomorphically onto JSJ tori of M. Then we can embed $\overline{M}$ in a graph manifold M' such that the partial cover p extends to a finite cover $M' \to M$.*

Proof. For each M_v let d_v be the degree of the (possibly disconnected) cover $p^{-1}(M_v) \to M_v$. Similarly, let $d_{v,w}$ be the degree of $p^{-1}(T) \to T$ for the torus $T = M_v \cap M_w$. Take any $D \geq \max_{v,w} d_{v,w}$. Take $D - d_v$ copies of M_v for each v. These copies together with $\overline{M}$ glue up to form the desired cover M'. $\square$

Construction 4.13. Assume that S is straight in M (see Corollary 4.3) and a number $N > 0$ is divisible by all the degrees of (possibly disconnected) covering maps $S \cap M_v \to F_v$. We will construct an S–injective cover M_N^S of M.

First consider a horizontal block M_v. Put $n = |S \cap M_v|$, which denotes the number of connected components of $S \cap M_v$. Let d be the degree of $S_0 \to F_v$ for a (hence any) component S_0 of $S \cap M_v$ (they are all parallel). For each S_0 we take the unique degree $\frac{N}{n}$ cover of M_v to which S_0 lifts. It is the quotient of the $M_v^{S_0}$ block of M^S (see Definition 4.9) by the $\frac{N}{n}$th power of the generator of covering transformations. The result of this over the boundary of M_v is that if there are k components of intersection of S with a boundary torus, then in the cover we get k tori components projecting with degree $\frac{N}{k}$. Hence for two adjacent horizontal blocks we have a matching between the elevations of the JSJ tori crossed by (the lifts of) S_0's.

We now consider a vertical block M_v. Fix a component S_0 of $S \cap M_v$. Let F'_v be the double cover of F_v determined by the $\mathbf{Z}_2$ cohomology class of a non-separating simple closed curve on F_v. Each boundary component C of F_v is covered in F'_v by a pair C_1, C_2. Suppose S_0 connects tori T, Q, with base boundary circles of F_v denoted by C^T, C^Q. Put $t = |S \cap T|, q = |S \cap Q|$. Pick disjoint embedded arcs θ, ω in F'_v joining the pair C_i^T and the pair C_i^Q. Take $\frac{N}{t} - 1$ extra copies of F'_v containing copies of θ and $\frac{N}{q} - 1$ extra copies of F'_v containing copies of ω. Cutting and regluing along these arcs in cyclic orders gives a cover of F_v whose all boundary components project homeomorphically, except two degree $\frac{N}{t}$ covers of C^T and two degree $\frac{N}{q}$ covers of C^Q. To get a cover of M_v we product with S^1.

We now take one such covering block for each component S_0 of $S \cap M_v$ for M_v vertical and two blocks described before for M_v horizontal. All boundary components match except that there are some hanging boundary components giving rise to cut tori.

This concludes the construction of a partial cover. (Note that the lift of S crosses all blocks of this partial cover that are covering vertical blocks of M and every other block covering a horizontal one.) This partial cover satisfies the hypothesis of Lemma 4.12. We use it to obtain a (non-unique) S–injective cover whose relation with N we will record by denoting it M_N^S.

5 Separability of a surface

Proof of Theorem 1.1. We can assume that M is simple (see Section 2) and S is straight (see Corollary 4.3).

If S is a vertical torus contained in a single block, then $\pi_1(S)$ is separable

by Corollary 4.5. Otherwise, S contains a horizontal piece. Let $\tilde{m}$, the basepoint of the universal cover $\widetilde{M}$ of M, be chosen in the interior of a horizontal piece of the elevation $\widetilde{S}$ of S to $\widetilde{M}$ stabilised by $\pi_1(S) \subset \pi_1(M)$. Let $g \in \pi_1(M)$ be an element outside $\pi_1(S)$ and let $\widetilde{\gamma}$ be a path in $\widetilde{M}$ representing g, i.e. joining $\tilde{m}$ to $g\tilde{m}$. Then $\widetilde{\gamma}$ does not end on $\widetilde{S}$. Our goal is to find a finite cover with the same property.

We can assume that $\widetilde{\gamma}$ crosses as few elevations of JSJ tori as possible (in other words, it does not "backtrack" in $\widetilde{M}$.) Denote by $\widetilde{B}$ the last non-empty block of $\widetilde{M}$ entered by $\widetilde{\gamma}$ (possibly all blocks crossed by $\widetilde{\gamma}$ are non-empty, which means that we take $\widetilde{B}$ to be the last one).

We first consider the case where $\widetilde{B}$ is indeed the last block of $\widetilde{M}$ entered by $\widetilde{\gamma}$. Then $\widetilde{B}$ is horizontal (by the choice of $\tilde{m}$). In the quotient $B^S \subset M^S$ of $\widetilde{B}$ the projection of the endpoint of $\widetilde{\gamma}$ is still disjoint from S and the same is true in a sufficiently large cyclic quotient of the block B^S. This quotient coincides with appropriate block of the cover M^S_N from Construction 4.13. Hence for N sufficiently large the cover M^S_N is as desired.

We now consider the case where $\widetilde{B}$ is not the last block entered by $\widetilde{\gamma}$. Note that then $\widetilde{B}$ is vertical. By Corollary 4.4 we can pass to a finite cover M' where the projection γ' of $\widetilde{\gamma}$ does not backtrack, i.e. it does not cross twice the same JSJ torus. (Note that we require not only two consecutively crossed JSJ tori to be distinct, but all of them to be distinct.) This property will be preserved under taking further covers.

We denote by S' the elevation of S to M'. For separability of $\pi_1(S)$ we must prove that γ' does not end in S', either already in M' or under passing to a further finite cover.

Let $\widetilde{T}$ denote the universal cover of a JSJ torus through which $\widetilde{\gamma}$ leaves $\widetilde{B}$. Let T' be its quotient in M'. Our first step is to guarantee that in M' (or its finite cover), the surface S' does not cross T' (like in $\widetilde{M}$).

The quotient block $B' \subset M'$ of $\widetilde{B}$ is vertical. Denote by p and q the first and the last point of the intersection of γ' with B'. Let S'_0 be the quotient in B' of $\widetilde{S} \cap \widetilde{B}$ (there might be some other components of $S' \cap B'$). Let K be the JSJ torus crossed by S'_0 other then the one containing p. Any further S'–injective cover (for example $M'^{S'}_N$ from Construction 4.13) satisfies our condition $S' \cap T' = \emptyset$ unless $q \in K$ (i.e. $T' = K$), see Figure 5. In that case we first, using Lemma 4.2, pass to a cover where (keeping the same notation) the point q does not lie in K. So S'_0 does not cross T'. Still, there might an accidental component of $S' \cap B'$ crossing T'. We can remove it by passing to an S'–injective cover.

Summarizing, we have constructed a cover M' where T' is disjoint from S'. It suffices now to pass to a degree 2 cover determined by the cohomology

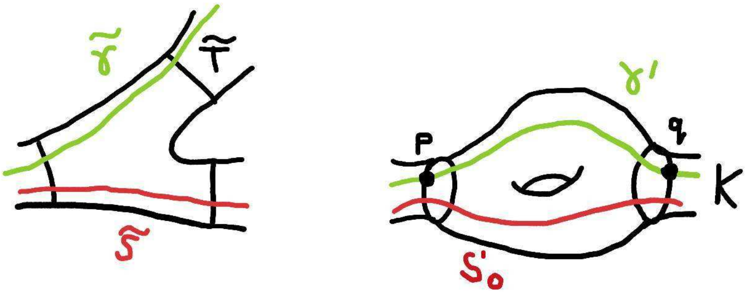

Figure 5: q requires removing from K

class $[T'] \in H^1(M', \mathbf{Z}_2)$. In that cover the portion of γ' after q is contained entirely in the union of empty blocks. In particular, its end lies outside the appropriate lift of S', as desired. □

6 Separability of crossing surfaces

Outline of the argument. Denote by $\widetilde{S}$ and $\widetilde{P}^0$ the elevations of S and P to the universal cover $\widetilde{M}$ of M determined by the embeddings $\pi_1(S), \pi_1(P) \subset \pi_1(M)$. By our hypothesis, $\widetilde{S} \cap \widetilde{P}^0$ is non-empty, hence we can choose the basepoint $\tilde{m} \in \widetilde{S} \cap \widetilde{P}^0$.

We fix $g \in \pi_1(M)$ outside $\pi_1(S)\pi_1(P)$ and take a path $\widetilde{\gamma}$ starting at $\tilde{m}$ representing g in $\widetilde{M}$. We denote by $\widetilde{P}$ the elevation of P through the terminal point $g\tilde{m}$ of $\widetilde{\gamma}$. We want to find a finite quotient of $\widetilde{M}$, where $\widetilde{S}$ and $\widetilde{P}$ are "disjoint" in the sense that they do not intersect at a basepoint-translate.

The main object we work with is the "core" $\overline{M} \subset \widetilde{M}$ consisting of blocks intersecting simultaneously $\widetilde{S}$ and $\widetilde{P}$. In Step 1 we prove that for each $\pi_1(S)$ orbit in $\overline{M}$ of a core block, we can quotient it to a finite block with "disjoint" quotients of $\widetilde{S}$ and $\widetilde{P}$.

In Step 2 we use that to show how to quotient simultaneously the whole core (in fact, its extension) to a finite quotient $\widehat{M}'$ where the images of $\widetilde{S}$ and $\widetilde{P}$ are "disjoint" (Step 2(i)). Moreover, we arrange that the images $\widetilde{S}$ and $\widetilde{P}$ never intersect simultaneously the same cut torus of the partial (see Definition 4.11) cover $\widehat{M}' \to M$ (Step 2(iii)). The partial cover $\widehat{M}'$ extends

to a finite cover M' by Step 2(ii).

Finally, in Step 3, we use Step 2(iii) to pass to a further cover, where it is possible for the quotients of $\widetilde{S}$ and $\widetilde{P}$ to meet only in the image of the core. But this is excluded by Step 2(i).

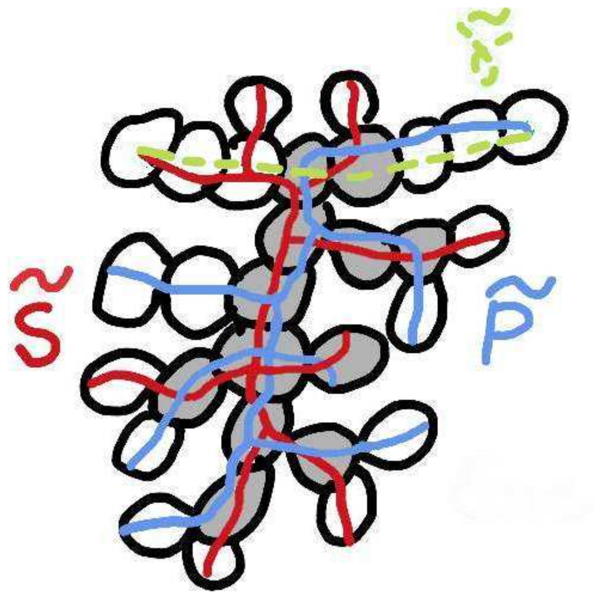

Figure 6: core of $\widetilde{M}$

Proof of Theorem 1.2. We choose $\tilde{m} \in \widetilde{S} \cap \widetilde{P}^0$ as in the outline of the argument. Without loss of generality, we can assume that if $\tilde{m}$ lies in a vertical piece of $\widetilde{P}^0$, then it also lies in a vertical piece of $\widetilde{S}$. Note that by Corollary 4.3 under passing to a finite cover we can assume that S and P are straight. As usual, M can be also assumed to be simple and S–injective.

Let $g \in \pi_1(M)$ be an element outside $\pi_1(S)\pi_1(P)$ and let $\widetilde{\gamma}$ be a a path representing g in $\widetilde{M}$ as in the outline. We can assume that $\widetilde{\gamma}$ goes through as few blocks of $\widetilde{M}$ as possible.

Recall that $\widetilde{S}$ is the elevation of S to $\widetilde{M}$ passing through $\tilde{m}$, where $\widetilde{\gamma}$ starts, and $\widetilde{P}$ is the elevation of P to $\widetilde{M}$ passing through $g\tilde{m}$, where $\widetilde{\gamma}$ terminates. Our hypothesis on g says that $\widetilde{S}$ and $\widetilde{P}$ do not cross at any translate of the basepoint $\widetilde{m}$ (we will shortly refer to such point or its quotient in an

intermediate cover as a *basepoint-translate*). Our separability goal is to find a finite cover of M with the same property.

The *core* $\overline{M}$ of $\widetilde{M}$ is the union of blocks intersecting both $\widetilde{S}$ and $\widetilde{P}$ (see Figure 6, note however that the core might consist of an infinite number of blocks). Assume that the core is non-empty, we will consider the other case at the end of the whole proof.

We put $\overline{S} = \widetilde{S} \cap \overline{M}$, $\overline{P} = \widetilde{P} \cap \overline{M}$. Let $\overline{M}^S$ be the manifold obtained from the core by identifying points in the same orbit of $\pi_1(S)$ (this is not a genuine action on $\overline{M}$, only a partial one). In this identification we exceptionally treat the core as an open manifold, so we do not identify boundary components whose adjacent core blocks are not identified.

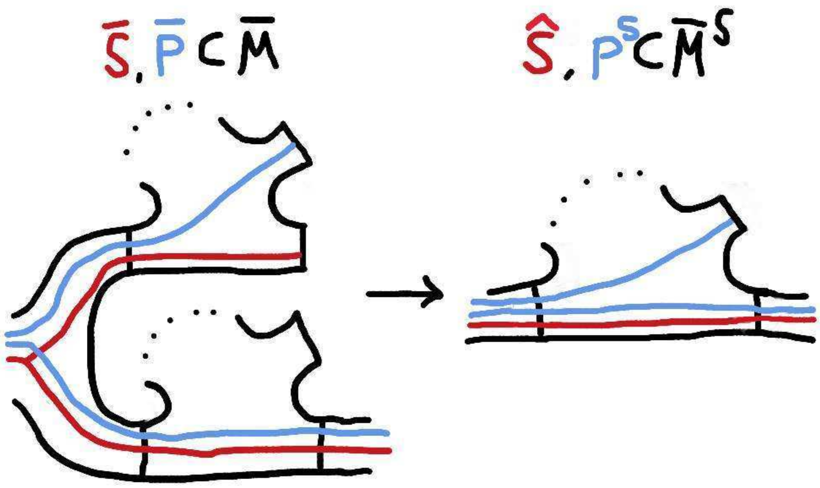

Figure 7: $\overline{P} \to P^S$ is proper

Let P^S be the quotient of $\overline{P}$ in $\overline{M}^S$ and let $\widehat{S}$ be the quotient of $\overline{S}$ in $\overline{M}^S$. Note that P^S and $\widehat{S}$ do not go through the same basepoint-translate. The notation $\widehat{S}$ instead of S^S is justified by the fact that $\widehat{S}$ is in fact a lift of a "core" subsurface of S. Also note that though the map $\overline{M} \to \overline{M}^S$ is not *proper* in the sense that a boundary component of $\overline{M}$ might be mapped into the interior $\overline{M}^S$, its restriction to $\overline{P} \to P^S$ is proper (see Figure 7). In other

words, boundary components of $\overline{P}$ are mapped onto boundary components of P^S.

Step 1. *Let B^S be a block in $\overline{M}^S$ covering a block B of M. Then B^S quotients to a finite cover B^* of B where the quotients of P^S and $\widehat{S}$ still do not intersect at a basepoint-translate.*

Loosely speaking, at least at a single core block we can achieve separability. Note that the property of the finite cover B^* in Step 1 is preserved under passing to a further cover which is a quotient of B^S.

First assume that $\tilde{m}$ lies in a horizontal piece of $\widetilde{P}^0$. Consequently, if a basepoint-translate lies in B^S, then the block B^S is P^S–horizontal. Denote $S_0 = \widehat{S} \cap B^S$. First consider the case where S_0 is vertical. Then there are only finitely many elevations of $P \cap B \subset M$ to B^S: their number is bounded by the degree of $P \cap B \to F$, where F is the base surface of B. The action of covering transformations of B on the universal cover of the block B^S factors to an action on B^S. A finite index subgroup preserves all the elevations of $P \cap B$, since there are only finitely many of them. We quotient by this subgroup to obtain a desired finite cover B^* of B.

Now consider the case where S_0 is horizontal. Then the action of covering transformations on the universal cover of the block B factors to $\mathbf{Z} = \langle c \rangle$ action on B^S. The easy subcase is where one (hence any) elevation of $P \cap B$ to B^S is non-compact (see Figure 8). Then like in the previous case there are only finitely many elevations of $P \cap B$ and we can choose a finite cover B^* coming from quotienting by a power subgroup $\langle c^k \rangle$ which maps $P^S \cap B^S$ onto itself.

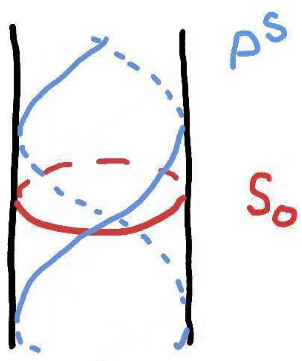

Figure 8: noncompact components of $P^S \cap B^S$: cross-section by a cylinder

The interesting subcase is where the elevations of $P \cap B$ to B^S are compact. Let P_0 be any component of $P^S \cap B^S$. Let $\mathcal{P}$ be the maximal connected subsurface of P^S containing P_0 consisting uniquely of horizontal pieces. First consider the situation where P_0 is properly contained in $\mathcal{P} \cap B^S$. We will show that $P^S \cap B^S$ is invariant under some c^k (like in the case of non-compact P_0).

Note that the pieces of $\mathcal{P} \cap B^S$ might not lie in one $\langle c \rangle$–orbit. However, we can extend this action to another cyclic action $\langle c \rangle \subset \langle \underline{c} \rangle$ on B^S by homeomorphisms for which all the pieces of $\mathcal{P} \cap B^S$ lie in one orbit.

Assume that for some $k \neq 0$ the translate $\underline{c}^k P_0$ lies in $\mathcal{P}$. Let $P_i \subset B_i$ be a sequence of pieces in $\mathcal{P}$ and their blocks in $\overline{M}^S$ connecting P_0 to $\underline{c}^k P_0$. The action of $\underline{c}$ extends to all blocks B_i crossed by $\mathcal{P}$ (some might be $\widehat{S}$–vertical). Hence for any n there is a sequence of pieces that are translates of P_i joining $\underline{c}^n P_0$ to $\underline{c}^{k+n} P_0$. Thus $\underline{c}^n P_0$ lies in P^S if and only if $\underline{c}^{k+n} P_0$ does, in other words $P^S \cap B$ is $\underline{c}^k$–invariant. Then for some k' we have that $P^S \cap B$ is $c^{k'}$–invariant, as desired.

It remains to consider the situation where P_0 equals $\mathcal{P} \cap B^S$. Then, by the c–action argument above, the same is true for any choice of P_0 in $P^S \cap B^S$. Moreover, since there are only finitely many vertical pieces of P^S in $\overline{M}^S$ with both boundary components in the interior of $\overline{M}^S$, only finitely many translate copies od $\mathcal{P}$ are joined together contributing to $P^S \cap B^S$. Concluding, $P^S \cap B^S$ is compact. Then for any sufficiently large k, no element of $\langle c^k \rangle$ maps a basepoint-translate in S_0 onto a point in P^S. This finishes the argument for Step 1 under the assumption that $\tilde{m}$ lies in a horizontal piece of $\widetilde{P}^0$.

Finally, assume that $\tilde{m}$ lies in the intersection of a vertical piece of $\widetilde{P}^0$ and a vertical piece of $\widetilde{S}$. Consequently, if a basepoint-translate lies in B^S, then B^S is both P^S– and $\widehat{S}$–vertical. Let K_1, K_2 be the boundary cylinders of B^S crossed by $\widehat{S}$. By the definition of the core, except for the exceptional situation where the core is a single block and hence $P^S \cap B^S$ has just one component, each piece of $P^S \cap B^S$ intersects some K_i. By the c–action argument applied to adjacent (P^S–horizontal) blocks of B^S, for each $i = 1, 2$, the intersection $P^S \cap K_i$ is either compact or periodic. Then after quotienting B^S by finite index subgroups of one, both or none of the stabilisers of K_i we obtain $\check{B}$, in which the quotient of P^S is compact and still does not intersect $\widehat{S}$ in a basepoint-translate. Let $\check{F} \to F$ be the cover induced between the base surfaces of $\check{B} \to B$. By separability of $\pi_1(\check{F})$ in $\pi_1(F)$, the cover $\check{B}$ quotients further to a desired cover B^*. This finishes the argument for Step 1.

Denote by $\widehat{M}$ the quotient of $\overline{M}$ (and hence $\overline{M}^S$) in M. However, if there is a JSJ torus K in M outside the image of the interior of $\overline{M}$ but both of its

adjacent blocks lie in the image of $\overline{M}$, then we put in $\widehat{M}$ two copies of K, each compactifying one of the adjacent blocks. In other words, $\widehat{M}$ is contained in M only in the sense of manifolds open at the boundary. Note that since M is S–injective, each block of $\widehat{M}$ is covered by exactly one block of $\overline{M}^S$. Keep in mind, however, that $\overline{M}^S \to \widehat{M}$ is only a partial infinite cover.

Step 2. *There is a finite cover $\widehat{M}'$ of $\widehat{M}$ through which the map $\overline{M} \to \widehat{M}$ factors, with the following properties. Let $\widehat{S}', \widehat{P}' \subset \widehat{M}'$ be the extensions of the quotients of $\overline{S}, \overline{P}$ in $\widehat{M}'$ to elevations of $S \cap \widehat{M}$ and a component of $P \cap \widehat{M}$.*

(i) Each block B' of $\widehat{M}'$ is a further cover of a cover B^* satisfying Step 1. Moreover, the images of $\widehat{S}' \cap B'$, $\widehat{P}' \cap B'$ in B^* are contained in the quotients of P^S and $\widehat{S}$.

(ii) Over all boundary tori the cover $\widehat{M}' \to \widehat{M}$ is $n!$ characteristic (for some common n). In other words, it corresponds to the subgroup $n!\mathbf{Z} \times n!\mathbf{Z} \subset \mathbf{Z} \times \mathbf{Z}$.

(iii) Each cut torus of the partial cover $\widehat{M}' \to M$ intersects at most one of $\widehat{S}', \widehat{P}'$.

Moreover, $\widehat{M}'$ is $\widehat{S}'$–injective.

Note that in view of Step 1, Step 2(i) implies immediately that $\widehat{S}'$ and $\widehat{P}'$ do not intersect at a basepoint-translate.

The value n is the maximum of n needed to execute the following construction over each of the finitely many blocks of $\widehat{M}$.

First suppose that $B \subset \widehat{M}$ is P–horizontal and S–horizontal. Take the cyclic cover B^* guaranteed by Step 1. It may be taken with any degree $n!$ for n sufficiently large. In order to make the quotient to B characteristic on the boundary tori, we pass from B^* to a cover B' induced by any cover of $S \cap B$ of degree $n!$ on each boundary component (use Lemma 4.7).

Now assume that $B \subset \widehat{M}$ is P–horizontal but S–vertical. Again take the cover B^* guaranteed by Step 1. By Lemma 4.7 it may be chosen to be degree $n!$ on boundary tori for n sufficiently large. In order to make the quotient to B characteristic on the boundary tori, we pass from B^* to a cyclic cover B' of degree $n!$ determined by an arbitrary degree 1 horizontal surface.

In the case where the block $B \subset \widehat{M}$ is both P–vertical and S–vertical finding convenient B^* will involve several steps. First of all, there is a cover B^* of B satisfying Step 1. Since we still want to replace B^* by a particular finite cover, in order to simplify the notation we will assume that already B

has the property from Step 1, that $S \cap P$ and $P \cap B$ do not intersect at a basepoint-translate. This property is invariant under taking covers, so any further cover B^* that we will construct will satisfy Step 1.

First assume that we are in a (simpler) subcase where in B^S (the block in $\overline{M}^S$ covering B) there is a vertical annular piece of P^S homotopic to $\widehat{S} \cap B^S$. In that case we use readily Proposition 4.6 with F being the base surface of the block B. Let $\alpha \subset F$ be the base arc $S \cap B$. Let $\mathcal{A}$ be the family of those base arcs of $P^S \cap B^S$ which intersect the same components of ∂F as α.

By Proposition 4.6, for sufficiently large n we get a finite quotient B^* of B^S of degree $n!$ on all boundary components over B. The base surface of the cover B^* satisfies also (*), which will be used later. Like before, in order to get a characteristic cover over the boundary tori, we pass to a cover B' of B^*, determined by an arbitrary element of $H^1(B^*, \mathbf{Z}_{n!})$ dual to a degree one horizontal surface in B^*.

In the (harder) subcase where in B^S there is no vertical annular piece of P^S homotopic to $\widehat{S} \cap B^S$, in Proposition 4.6 we want to use, instead of B, an intermediate finite quotient B_{int} of B^S which also has the property that the projection of the vertical annular piece of $\widehat{S}$ is not homotopic to the projection of any piece of P^S. (B might not have this property.)

Let K_1, K_2 be the boundary cylinders of B^S crossed by $\widehat{S}$. By the argument of Step 1 applied to adjacent blocks of B^S, for each $i = 1, 2$ either $P^S \cap K_i$ is compact, or there is a subgroup $k_i\mathbf{Z} \subset \mathbf{Z}$ of the covering transformations of K_i that preserves $P^S \cap K_i$. This proves that any finite quotient B_{int} of B^S such that $B_{int} \to B$ over the quotients of K_i in B is of degree sufficiently high and divisible by k_1k_2 is as required. Now we repeat the construction of B^* using Proposition 4.6 with B_{int} in place of B. In other words the surface F is the base surface of B_{int}. However, we require that the boundary degrees are $n!$ when we quotient B^* to B, not to B_{int}. Again, we record property (*) for the later use. We obtain B' from B^* as before. This closes the discussion of the case, where B is both P–vertical and S–vertical.

Here is the diagram illustrating the intermediate covers between the universal $\overline{B} \to B$.

$$\begin{tikzcd}
\overline{B} \arrow[r] \arrow[d] & B' \arrow[d] & B_{int} \arrow[dr] & \\
B^S \arrow[r] & B^* \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

The last case is where a block B is P–vertical and S–horizontal. Here n is arbitrary. We first take the degree $n!$ cyclic cover B^* of B determined

by $[S_0] \in H^1(B^*, \mathbf{Z}_{n!})$, where $S_0 = S \cap B$. Next we pass to a cover B' induced by any cover $S_0' \to S_0$ of degree $n!$ on each boundary component (use Lemma 4.7).

We take n sufficiently large for both the construction in Step 1 and various applications of Proposition 4.6 with the above data to work. Then all blocks B have covers B' that are $n!$ characteristic on the boundary. Now we take the right number of copies of each of B' so that the degree of the disconnected cover from the union of the copies of B' to B does not depend on B. In each of these copies we distinguish one elevation Σ_c of (connected) $S \cap B$. In the case where B is S–horizontal, the surface Σ_c is of degree 1 over the base surface of B'. Hence the intersection of Σ_c with each boundary torus is at most a single curve. We match up these blocks, also matching the Σ_c's, to form a cover $\widehat{M}'$ of $\widehat{M}$. We pick any of the maps $\overline{M} \to \widehat{M}'$ mapping $\overline{S}$ to the union Σ of the Σ_c's. We will now verify that all the required properties of $\widehat{M}'$ hold.

Property (ii) is clear from the construction. Note that $\widehat{M}'$ is $\widehat{S}'$–injective, since $\widehat{S}'$ is contained in Σ. Moreover, obviously for each copy B_c in $\widehat{M}'$ of a block B' which covers the quotient B^* of B^S in $\overline{M}^S$ we have the following. Under the identification of B_c with B', the projection to B^* of the intersection $\widehat{S}' \cap B_c$ is contained in the projection to B^* of $\widehat{S} \cap B^S$. We now claim the same for $\widehat{P}'$: the projection to B^* of the intersection $\widehat{P}' \cap B_c$ is contained in the projection to B^* of $P^S \cap B^S$.

Before we justify the claim, we point out that the issue here is that although the map $\overline{M} \to \widehat{M}$ factors through $\widehat{M}'$, the image of $\overline{P}$ in $\widehat{M}'$ might be smaller than $\widehat{P}'$. So a priori it is not clear, where the extension $\widehat{P}'$ is located. We look for a surface Π for $\widehat{P}'$ which replaces Σ for $\widehat{S}'$ in the argument above.

Also note that the claim together with the fact that B^* was chosen as in Step 1 implies property (i).

To justify the claim, let H be the set of elements $h \in \pi_1(S)$ preserving that copy of the universal cover of $\widehat{M}'$ in $\widetilde{M}$ which contains $\overline{M}$. In other words, $H = \pi_1(\widehat{S})$. (Note that the cosets $H/\pi_1(\widehat{S}')$ correspond to different ways in which we could have defined the projection $\overline{M} \to \widehat{M}'$ above.) Let $\Pi \subset \widehat{M}'$ be the union of the projections of $h\overline{P}$, over $h \in H$. Obviously Π has the property that for each $B_c \simeq B'$ as above the projection to B^* of the intersection $\Pi \cap B_c$ is contained in the projection of $P^S \cap B^S$. Since the projection of $\overline{P}$ is contained in Π, in order to justify the claim, it remains to prove that Π is a surface properly embedded in $\widehat{M}'$:

The quotient in $\widehat{M}'$ of specified translate $h\overline{P}$ with $h \in H$ might fail to

be proper only at a quotient of a boundary line $h\pi$ of $h\overline{P}$. If π lies in $\overline{M}$ in the boundary of a $\overline{P}$–horizontal and $\overline{S}$–vertical block, then the quotient of $h\pi$ lies also in the boundary of $\widehat{M}'$ (since it is $\widehat{S}'$–injective). The only other possibility is that π lies in the boundary of a $\overline{P}$–vertical and $\overline{S}$–vertical block $\overline{B}$: Denote by ψ the opposite boundary line in $\overline{B}$ of the piece of $\overline{P}$ containing π. Also, denote by σ the boundary line of $\overline{S} \cap \overline{B}$ in the plane not containing ψ. The only boundary component of the quotient of $h\overline{B}$ in $\widehat{M}'$ which is possibly in the interior of $\widehat{M}'$, besides the one containing the quotient of $h\psi$, is the one containing the quotient of $h\sigma$. Assume then that $h\pi$ and $h\sigma$, hence also π and σ are mapped into the same JSJ torus of $\widehat{M}'$. Property (*) of Proposition 4.6 says then that we were in the "simpler subcase" of the discussion above: in B^S there is a vertical piece p of P^S homotopic to $\widehat{S} \cap B^S$. Moreover, the vertical annulus p contains π upon passing to the quotient B^* guaranteed in Proposition 4.6. See Figure 9. Let $f\overline{B}$ be the block of $\overline{M}$ from which p arises, with $f \in H$. In $f\overline{B}$ the pieces of $\overline{S}, \overline{P}$ are also homotopic. By the definition of the core, the line $f\pi$ is in the interior of $\overline{P}$. Hence $h\pi$ is in the interior of $hf^{-1}\overline{P} \subset \Pi$. This finishes the argument for the claim and hence for property (i).

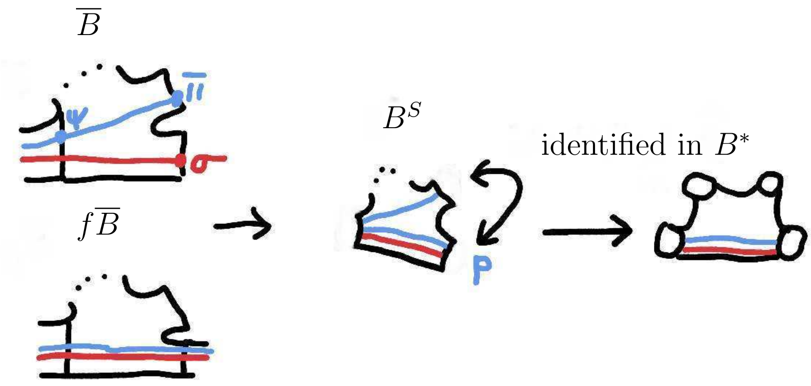

Figure 9: finding p

As for property (iii), we also need to use the conclusion (*) of Proposition 4.6. Let K' be a cut torus of $\widehat{M}'$ in a copy of a block B'. Let $B^S \subset \overline{M}^S$ be the block mapped to the same B^* as B' and let K^S be that elevation from B^* to B^S of the quotient of K', which crosses $\widehat{S}$. Then K^S lies also in the

boundary of $\overline{M}^S$. Hence P^S is disjoint from K^S by the definition of the core. In view of (*) in the "harder subcase", the quotient of $\overline{P}$ in $\widehat{P}'$ is disjoint from K'. The same is true for $h\overline{P}$ over $h \in H$ (H as in the proof of the claim above), hence for the whole $\widehat{P}'$. Thus we have proved property (iii), that $\widehat{S}'$ and $\widehat{P}'$ do not cross K' simultaneously. This completes the argument for Step 2.

The graph manifold $\widehat{M'}$ is just a partial cover of M. By Step 2(ii) we can complete it to a genuine cover M' by taking appropriate number of disjoint copies of any finite covers of blocks in M that are $n!$ characteristic on the boundary. We require that $\widehat{M'}$ embeds in M' as a closed submanifold — we do not allow accidental matching of boundary components of open $\widehat{M'}$. By choosing those covers correctly we keep M' to be S'–injective, where $S' \subset M'$ is the appropriate elevation of S. It remains to perform the following.

Step 3. *There is a finite cover M'' of M', whose all blocks B'' intersecting simultaneously the quotients S'', P'' of $\widetilde{S}$ and $\widetilde{P}$ project to $B' \subset \widehat{M'}$ so that $S'' \cap B''$ maps into $\widehat{S}'$ and $P'' \cap B''$ maps into $\widehat{P}'$.*

Let $\check{M}$ be a P'–injective cover of M'. We keep the notation P' for the lift of P' to $\check{M}$ (quotient of $\widetilde{P}$) and denote by $\check{S}$ the appropriate elevation of S (quotient of $\widetilde{S}$). Let τ be the union of the JSJ tori of $\check{M}$ containing the boundary components of $\widehat{P}'$ which are not in the boundary of P'.

We consider the degree 2 cover M'' of $\check{M}$ defined by the $\mathbf{Z}_2$ cohomology class $[\tau]$. The union of tori τ is disjoint from $\check{S}$, by Step 2(iii). On the other hand, by P'–injectivity, τ separates P' into $\widehat{P}'$ and its complement. Hence both $\check{S}$ and P' lift to M'' and any piece of lifted $P' \setminus \widehat{P}'$ is in an $\check{S}$–empty block of M''. This implies that the cover M'' satisfies Step 3.

Conclusion. By Step 3, if surfaces S'' and P'' intersect in a block B'' of M'', then they project to surfaces $\widehat{S}'$ and $\widehat{P}'$ in a block B' of $\widehat{M'}$. By Step 2(i), the surfaces $\widehat{S}'$ and $\widehat{P}'$ do not intersect in a basepoint-translate. Then the same is true for S'' and P''.

This concludes the proof of the main theorem, up to discussing the case, where the core is empty. Then first, using Corollary 4.4 we pass to a cover M, where the path γ representing g does not go twice through the same block. By possibly passing to a further cover we also assume that M is S–injective. Then instead of the core we consider the minimal connected graph submanifold of $\widetilde{M}$ crossed by both $\widetilde{S}$ and $\widetilde{P}$. Its blocks are in correspondence with some of the blocks of M crossed by γ. Steps 1 and 2 are now immediate and we perform Step 3 as in the main argument. □

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